Calculation of Eigenvector Derivatives for Structures with Repeated Eigenvalues

William C. Mills-Curran*
Sandia National Laboratories, Albuquerque, New Mexico

In structural optimization and system identification, eigenvector derivatives provide important information for updating design/model parameters. When the current parameter values yield repeated eigenvalues, it has not been possible previously to calculate unique eigenvector derivatives. Recent work has provided a method for determining unique eigenvalue derivatives for this case, but methods for calculating the eigenvector sensitivities have been incomplete. In this work, a complete method for calculation of repeated-root eigenvector derivatives is shown for the real, symmetric structural eigenproblem. The derivation is completed by using information from the second derivative of the eigen problem and is limited to the case of distinct eigenvalue sensitivities. As an example, the repeated-root eigenvector sensitivities are calculated for a simple three degree-of-freedom beam grillage. Comparisons of linear approximations (using these derivatives) to the calculated eigenvectors support the accuracy of the formulation.

Nomenclature

A	= eigenvector coefficient matrix
a_i	= ith column of the eigenvector coefficient matrix
C	= eigenvector sensitivity coefficient matrix
c_i	= ith column of the eigenvector sensitivity coefficient matrix
c_{ij}	= element of the eigenvector sensitivity coefficient matrix
F_i	= eigenproblem matrix $(K-\lambda_i M)$ for the ith eigenvalue
K	= stiffness matrix (symmetric)
M	= mass matrix (symmetric)
N	= number of repeated eigenvalues
p	= order of the system
V_i	= nonhomogeneous portion of the ith eigenvector sensitivity
β	= design parameter
δ_{ij}	= Kronecker delta
λ,λ_i	= eigenvalue and element of eigenvalue vector, respectively
$\phi_i,\bar{\phi}_i,\tilde{\phi}_i$	= eigenvector for the ith eigenvalue
$\Phi,\bar{\Phi}$	= matrices of eigenvectors with repeated eigenvalues
$(\ )'$	= first-order derivative with respect to a design parameter
$(\ )''$	= second-order derivative with respect to a design parameter

I. Introduction

THE real, symmetric structural eigenproblem (defining K and M as the stiffness and mass matrices, respectively) is

$$(K-\lambda M)\phi=0 \tag{1}$$

with mass orthonormalization

$$\phi_i^T M\phi_i=1 \tag{2}$$

$$\phi_j^T M\phi_i=0, \qquad i\neq j \tag{3}$$

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*Member of Technical Staff, Division 1524. Member AIAA.

When the solution to Eq. (1) produces N repeated eigenvalues

$$\lambda_i=\lambda_j, \qquad i,j=1,\ldots,N \tag{4}$$

then computation of derivatives of the eigenvalues and eigenvectors is not straightforward. The complication is related to the fact that the eigenvectors Φ of the repeated eigenvalues are not unique. In fact, an infinite number of linear combinations of the eigenvectors will satisfy Eqs. (1), (2), and (3). Thus, a new eigenvector $\tilde{\phi}_i$ is a linear combination of the originally calculated eigenvectors ϕ_i, where the coefficient vector a_i is to be determined.

$$\Phi=[\phi_i\,|\phi_2|\ldots|\phi_N] \tag{5}$$

$$\tilde{\phi}_i=\Phi a_i, \qquad i=1,\ldots,N \tag{6}$$

Substitution of $\tilde{\phi}$ for ϕ in Eq. (1) shows that $\tilde{\phi}_i$ satisfies the original eigenproblem for any selection of a_i.

$$(K-\lambda_i M)\Phi a_i=0, \qquad i=1,\ldots,N \tag{7}$$

Equations (2) and (3) impose some constraints on a_i. Using the Kronecker delta function δ_{ij},

$$\delta_{ij}=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases} \tag{8}$$

Eqs. (2), (3), (6), and (8) yield

$$a_i^T\Phi^T M\Phi a_j=a_i^T a_j=\delta_{ij}, \qquad i,j=1,\ldots,N \tag{9}$$

Although Eq. (9) imposes limitations on the selection of a_i, any orthonormal set will satisfy these restrictions.

Denoting first and second derivatives with respect to a parameter by $'$ and $''$, the standard equation for calculating eigenvalue derivatives[1] results in a nonunique value for λ_i' due to the nonuniqueness of $\tilde{\phi}_i$ and a_i.

$$\lambda_i'=\tilde{\phi}_i^T(K'-\lambda_i M')\tilde{\phi}_i, \qquad i=1,\ldots,N \tag{10}$$

Work by Lancaster[2] and more recently Chen and Pan[3] provide a technique for calculating the sensitivities of repeated eigenvalues, and Ojalvo[4,5] presents a partial solution to the eigenvector derivative problem using Nelson's method,[6] but a complete solution has not been presented.

In this paper, the solutions of Refs. 1-4 are reviewed for the repeated-eigenvalue problem as stated in Eqs. (1-4), and a complete method for calculating eigenvector derivatives is presented. This method uses the second derivative of the eigenvalue problem and treats the case where the repeated eigenvalues have distinct sensitivities.

II. Eigenvalue Derivatives

Reference 3 shows that eigenvalue derivatives for repeated roots can be found by posing a subeigenanalysis problem. For convenience, define

$$F_i \equiv K - \lambda_i M, \qquad i=1,\ldots,N \tag{11}$$

Using the nonunique eigenvectors $\tilde{\phi}$, the eigenproblem of Eqs. (1-3) becomes

$$F_i \tilde{\phi}_i = 0, \qquad i=1,\ldots,N \tag{12}$$

The derivative of Eq. (12) is taken, i.e.,

$$F_i' \tilde{\phi}_i + F_i \tilde{\phi}_i' = 0, \qquad i=1,\ldots,N \tag{13}$$

Substituting from Eq. (6) and premultiplying by Φ^T gives

$$\Phi^T F_i' \Phi a_i + \Phi^T F_i \tilde{\phi}_i' = 0, \qquad i=1,\ldots,N \tag{14}$$

Since $\Phi^T F_i$ is the transpose of the original eigenproblem solution, Eq. (14) becomes

$$\Phi^T F_i' \Phi a_i = 0, \qquad i=1,\ldots,N \tag{15}$$

Expanding Eq. (15)

$$[\Phi^T(K' - \lambda_i M')\Phi - \lambda_i' \Phi^T M \Phi] a_i = 0, \qquad i=1,\ldots,N \tag{16}$$

Because of Eqs. (2) and (3),

$$[\Phi^T(K' - \lambda_i M')\Phi - \lambda_i' I] a_i = 0, \qquad i=1,\ldots,N \tag{17}$$

Thus, a subeigenproblem in λ_i' and a_i is formed, producing a solution vector λ' and matrix A. Assuming that the λ' are distinct, a unique coefficient matrix A is generated that can be used with Eq. (6) to define unique eigenvectors $\bar{\Phi}$ and $\bar{\phi}$.

$$\bar{\phi}_i = \Phi a_i, \qquad i=1,\ldots,N \tag{18}$$

and

$$\bar{\Phi} = \Phi A \tag{19}$$

Note that if the eigenvalue derivatives are not distinct, then nonunique, although valid, values of the coefficient matrix A are generated. The special case of nondistinct eigenvalue derivatives is not addressed in this paper.

III. Eigenvector Derivatives

References 3 and 4 have provided partial guidance for calculating eigenvector derivatives in the presence of repeated eigenvalues; however, neither provides the complete solution. This section reviews the derivations previously published, then presents a new method that completes the calculation of the eigenvector sensitivities.

Stating Nelson's method[6] as extended by Ojalvo[4,5] for repeated roots, rearrange Eq. (13) and use the solutions of Eqs. (17), (18), and (19).

$$F_i \bar{\phi}_i' = -F_i' \bar{\phi}_i, \qquad i=1,\ldots,N \tag{20}$$

The eigenvector sensitivity $\bar{\phi}_i'$ is assumed to have the form

$$\bar{\phi}_i' = V_i + \bar{\Phi} c_i, \qquad i=1,\ldots,N \tag{21}$$

Note that because $\bar{\Phi}$ is the eigenvector matrix for the repeated eigenvalues,

$$F_i \bar{\Phi} = 0, \qquad i=1,\ldots,N \tag{22}$$

Substituting Eqs. (21) and (22) into Eq. (20) gives

$$F_i V_i = -F_i' \bar{\phi}_i, \qquad i=1,\ldots,N \tag{23}$$

Because F_i is of order p and rank $(p-N)$, it cannot be inverted. However, if the appropriate N rows and columns are eliminated from F_i along with the corresponding rows from the right-hand side of Eq. (23), then a valid solution for V_i can be found. Guidance for the partitioning of Eq. (23) is found in Ref. 4; however, this method may fail in some circumstances. A more rigorous approach is shown here.

Beginning with the original eigenproblem for the repeated roots and noting that any linear combination of the eigenvectors will solve the eigenproblem,

$$F_i \bar{\Phi} d_i = 0, \qquad i=1,\ldots,N \tag{24}$$

Equation (24) is partitioned so that $\bar{\Phi}^B$ and F_i^{BB} are square $N \times N$ matrices. (Reordering of rows and columns, as necessary, is not explicitly shown.)

$$\left[\begin{array}{c|c} F_i^{AA} & F_i^{AB} \\ \hline F_i^{BA} & F_i^{BB} \end{array}\right] \left[\begin{array}{c} \bar{\Phi}^A \\ \hline \bar{\Phi}^B \end{array}\right] d_i = 0, \qquad i=1,\ldots,N \tag{25}$$

Because the columns of $\bar{\Phi}$ are linearly independent, if a nonzero d_i can produce

$$\bar{\Phi}^B d_i = 0 \tag{26}$$

then

$$\bar{\Phi}^A d_i \neq 0 \tag{27}$$

Rewrite the upper portion of Eq. (25) as

$$F_i^{AA} \bar{\Phi}^A d_i = -F_i^{AB} \bar{\Phi}^B d_i, \qquad i=1,\ldots,N \tag{28}$$

If $\bar{\Phi}^B$ is singular, then, choosing d_i such that Eq. (26) is true, Eq. (28) becomes

$$F_i^{AA} \bar{\Phi}^A d_i = 0 \tag{29}$$

From Eqs. (27) and (29), it is seen that F_i^{AA} must be singular. Conversely, if $\bar{\Phi}^B$ is nonsingular, then the choice of the N elements of the d_i vector must uniquely determine the product of $\bar{\Phi}^A d_i$ because the system is of rank $(p-N)$, which is true [by Eq. (28)] only if F_i^{AA} is nonsingular. Thus, the rows and columns of F_i that are chosen for partitioning into F_i^{AA} should be selected so that $\bar{\Phi}^B$ is well conditioned.

Equation (23) is partitioned in the same manner as Eq. (25), and the upper portion is examined.

$$F_i^{AA} V_i^A = -F_i^{AB} V_i^B - (F_i' \bar{\phi}_i)^A, \qquad i=1,\ldots,N \tag{30}$$

where V_i^B is arbitrary and is selected to be zero.

$$F_i^{AA} V_i^A = -(F_i' \bar{\phi}_i)^A, \qquad i=1,\ldots,N \tag{31}$$

Because F^{AA} is nonsingular, Eq. (31) provides the solution for V_i^A and

$$V_i = \left\{ \frac{V_i^A}{0} \right\} \tag{32}$$

Reference 4 (also Ref. 6 for nonrepeated eigenvalues) indicates that c_i may be determined by differentiating the orthonormalization Eqs. (2) and (3).

$$\bar{\phi}_j^{T\,\prime} M\bar{\phi}_i + \bar{\phi}_j^T M'\bar{\phi}_i + \bar{\phi}_j^T M\bar{\phi}_i' = 0, \qquad i,j=1,\ldots,N \tag{33}$$

Substituting for $\bar{\phi}_i'$ from Eq. (21) and noting that the eigenvectors are orthonormal with respect to M,

$$(V_j + \bar{\Phi}c_j)M\bar{\phi}_i + \bar{\phi}_j^T M'\bar{\phi}_i + \bar{\phi}_j^T M(V_i + \bar{\Phi}c_i) = 0 \qquad i,j=1,\ldots,N \tag{34}$$

$$c_{ij} + V_j^T M\bar{\phi}_i + \bar{\phi}_j^T M V_i + \bar{\phi}_j^T M'\bar{\phi}_i + c_{ji} = 0 \qquad i,j=1,\ldots,N \tag{35}$$

Thus, for $i=j$,

$$c_{ii} = -\bar{\phi}_i^T(\tfrac{1}{2}M'\bar{\phi}_i + MV_i), \qquad i=1,\ldots,N \tag{36}$$

For $i \neq j$, Eq. (35) does not provide a unique solution. Thus, it is seen that differentiation of the orthonormalization equations can only provide the diagonal elements of C (the matrix of column vectors c_i). Reference 5 suggests a solution to this indeterminacy by introducing an additional arbitrary relationship.

IV. Off-Diagonal Terms

A new method is now shown that will allow calculation of the off-diagonal terms of C. It is noted that unique eigenvectors exist in the presence of distinct eigenvalue sensitivities [see Eq. (17)]. Similarly, eigenvector derivatives are linked to the second-order eigenvalue derivative. Thus, the second-order eigenvalue sensitivity is used to calculate the full eigenvector sensitivity by supplying the off-diagonal elements of the C matrix.

Equation (13) is differentiated after substituting $\bar{\phi}_i$ for $\tilde{\phi}_i$.

$$F_i'\bar{\phi}_i + F_i\bar{\phi}_i' = 0, \qquad i=1,\ldots,N \tag{37}$$

$$F_i''\bar{\phi}_i + 2F_i'\bar{\phi}_i' + F_i\bar{\phi}_i'' = 0, \qquad i=1,\ldots,N \tag{38}$$

Premultiplying by $\bar{\phi}_j^T$ gives

$$\bar{\phi}_j^T F_i''\bar{\phi}_i + 2\bar{\phi}_j^T F_i'\bar{\phi}_i' + \bar{\phi}_j^T F_i\bar{\phi}_i'' = 0, \qquad i,j=1,\ldots,N \tag{39}$$

From Eq. (4) and the symmetry of the matrices

$$\bar{\phi}_j^T F_i = (F_j\bar{\phi}_j)^T = 0, \qquad i,j=1,\ldots,N \tag{40}$$

Thus, the last term in Eq. (39) vanishes. Substituting for $\bar{\phi}_i'$ from Eq. (21) into Eq. (39) gives

$$\bar{\phi}_j^T F_i''\bar{\phi}_i + 2\bar{\phi}_j^T F_i'(V_i + \bar{\Phi}c_i) = 0, \qquad i,j=1,\ldots,N \tag{41}$$

$$\bar{\phi}_j^T F_i''\bar{\phi}_i + 2\bar{\phi}_j^T F_i' V_i + 2\bar{\phi}_j^T F_i'\bar{\Phi}c_i = 0, \qquad i,j=1,\ldots,N \tag{42}$$

The first and third terms of Eq. (42) are examined separately.

Term 1 (T_{ji}^1):

$$T_{ji}^1 \equiv \bar{\phi}_j^T F_i''\bar{\phi}_i, \qquad i,j=1,\ldots,N \tag{43}$$

Expand F_i''.

$$T_{ji}^1 = \bar{\phi}_j^T(K'' - 2\lambda_i' M' - \lambda_i M'' - \lambda_i'' M)\bar{\phi}_i, \quad i,j=1,\ldots,N \tag{44}$$

Since the eigenvectors are orthonormal with respect to M,

$$T_{ji}^1 = \bar{\phi}_j^T(K'' - 2\lambda_i' M' - \lambda_i M'')\bar{\phi}_i - \lambda_i''\delta_{ij}, \quad i,j=1,\ldots,N \tag{45}$$

Term 3 (T_{ji}^3):

$$T_{ji}^3 = 2\bar{\phi}_j^T F_i'\bar{\Phi}c_i, \qquad i,j=1,\ldots,N \tag{46}$$

Expand F_i'.

$$T_{ji}^3 = 2\bar{\phi}_j^T(K' - \lambda_i M' - \lambda_i' M)\bar{\Phi}c_i, \qquad i,j=1,\ldots,N \tag{47}$$

Because the eigenvalues are repeated,

$$T_{ji}^3 = 2\bar{\phi}_j^T(K' - \lambda_j M' - \lambda_j' M)\bar{\Phi}c_i + 2(\lambda_j' - \lambda_i')\bar{\phi}_j^T M\bar{\Phi}c_i \qquad i,j=1,\ldots,N \tag{48}$$

Noting the definition of F_j' and that the eigenvectors are orthonormal with respect to M,

$$\bar{\phi}_j^T M\bar{\Phi}c_i = c_{ji}, \qquad i,j=1,\ldots,N \tag{49}$$

$$T_{ji}^3 = 2\bar{\phi}_j^T F_j'\bar{\Phi}c_i + 2(\lambda_j' - \lambda_i')c_{ji}, \qquad i,j=1,\ldots,N \tag{50}$$

Equation (23) is premultiplied by $\bar{\Phi}^T$, i.e.,

$$\bar{\Phi}^T F_j V_j = -\bar{\Phi}^T F_j'\bar{\phi}_j, \qquad j=1,\ldots,N \tag{51}$$

Because $\bar{\Phi}$ is the matrix of eigenvectors,

$$\bar{\Phi}^T F_j V_j = 0 = -\bar{\Phi}^T F_j'\bar{\phi}_j = -(\bar{\phi}_j^T F_j'\bar{\Phi})^T, \quad j=1,\ldots,N \tag{52}$$

Substituting Eq. (52) into Eq. (50),

$$T_{ji}^3 = 2(\lambda_j' - \lambda_i')c_{ji}, \qquad i,j=1,\ldots,N \tag{53}$$

Recombining Eqs. (42), (45), and (53),

$$\bar{\phi}_j^T(K'' - 2\lambda_i' M' - \lambda_i M'')\bar{\phi}_i - \lambda_i''\delta_{ij} + 2\bar{\phi}_j^T F_i' V_i + 2(\lambda_j' - \lambda_i')c_{ji} = 0, \qquad i,j=1,\ldots,N \tag{54}$$

To find the off-diagonal elements of C, require $i \neq j$ and recall that the eigenvalue derivatives are assumed to be distinct.

$$c_{ji} = \frac{\bar{\phi}_j^T(K'' - 2\lambda_i' M' - \lambda_i M'')\bar{\phi}_i + 2\bar{\phi}_j^T F_i' V_i}{2(\lambda_i' - \lambda_j')} \qquad i,j=1,\ldots,N, \quad i \neq j \tag{55}$$

It is interesting to note that these c_{ji} as defined in Eq. (55) satisfy Eq. (35). Incidentally, if $i=j$ in Eq. (54), then the second derivative of the eigenvalue is found.

$$\lambda_i'' = \bar{\phi}_i^T(K'' - 2\lambda_i' M' - \lambda_i M'')\bar{\phi}_i + 2\bar{\phi}_i^T F_i' V_i \qquad i=1,\ldots,N \tag{56}$$

V. Example

A simple three degree-of-freedom (DOF) problem is used as an illustrative example. The planar grillage of Fig. 1 is composed of four DOF, consistent mass beam elements. These elements have no axial or torsional stiffness or mass. See Table 1 for the design data.

Figure 2 is a plot of two of the eigenvalues of the structure as a function of the design variable β, as β varies from 0.6 to 1.5 cm. Note that eigenvalues 1 and 2 cross at $\beta = 1$ cm. At this design,

$$\bar{\Phi} = \begin{bmatrix} 5.00 & 0.00 & 0.00 \\ 0.00 & 5.00 & 0.00 \\ 0.00 & 0.00 & 0.31 \end{bmatrix} \tag{57}$$

Table 1 Example problem parameters

Young's Modulus	3.0×10^7 nt/cm^2
Density	0.074 kg/cm^3
L	72 cm
I_1, I_2	$\beta^4/12$
I_3, I_4	(1/12) cm^4
A_1	β^2
A_2, A_3, A_4	1 in.^{2}
β_{initial}	1 in.

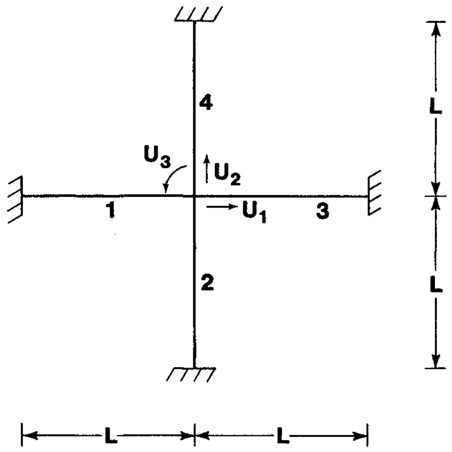

Fig. 1 Three degree-of-freedom structure.

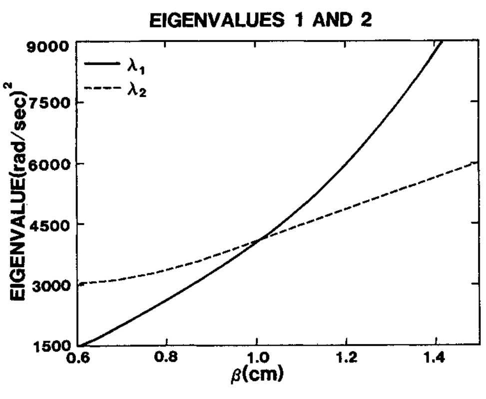

Fig. 2 Example problem: eigenvalues 1 and 2.

$$\lambda = \begin{Bmatrix} 4060 \\ 4060 \\ 52780 \end{Bmatrix} \tag{58}$$

Equations (17), (18), and (19) are applied to the problem.

$$\begin{Bmatrix} \lambda_1' \\ \lambda_2' \end{Bmatrix} = \begin{Bmatrix} 8120 \\ 4060 \end{Bmatrix} \tag{59}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{60}$$

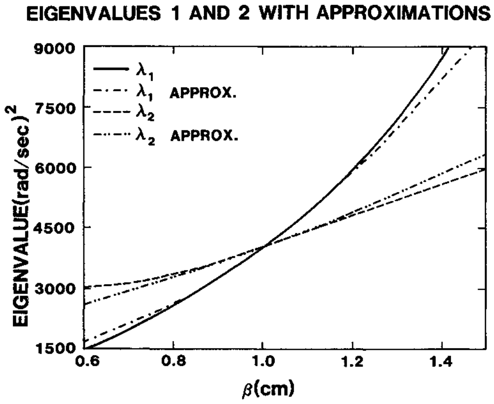

Fig. 3 Eigenvalue with second-order approximations.

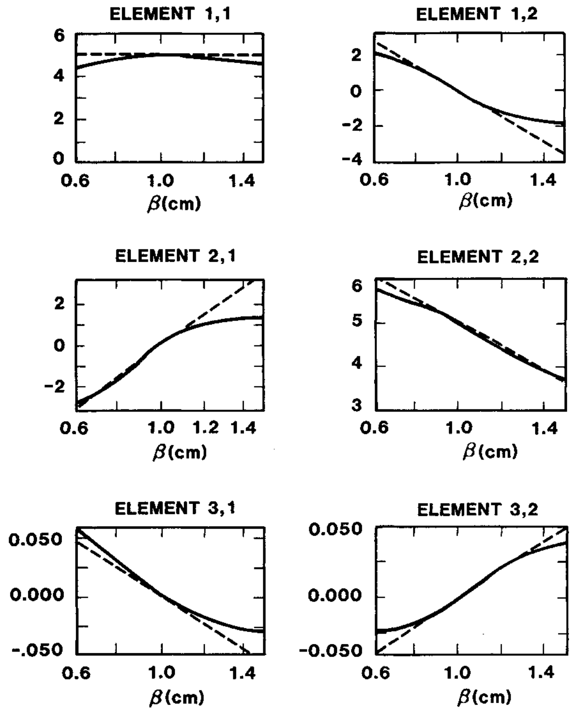

Fig. 4 Eigenvector elements with first-order approximations.

$$\bar{\Phi} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} \tag{61}$$

In this particular problem, the unique eigenvectors of Eq. (61) are the same as the trial eigenvectors of Eq. (57). This is due to the extreme simplicity of the example. At $\beta = 1$, the K and M matrices are diagonal and naturally give rise to the unique eigenvectors. For more complex problems, especially when a reduced number of eigenvalues and eigenvectors are calculated, the trial eigenvectors are different than the unique eigenvectors of Eq. (19).

A second-order approximation of λ_1 and λ_2 is formed using the sensitivities calculated with Eqs. (17) and (56). These second-order approximations are plotted in Fig. 3 along with the independently calculated eigenvalues of Fig. 2. Figure 3 shows that the calculated sensitivities are accurate.

The repeated-root eigenvector sensitivities are calculated using Eqs. (21), (23), (36), and (55).

$$[V_1 | V_2] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.113 & 0.0974 \end{bmatrix} \tag{62}$$

$$C = \begin{bmatrix} 0 & -1.4 \\ 1.4 & -0.5 \end{bmatrix} \tag{63}$$

$$\bar{\Phi}' = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.113 & 0.0974 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1.4 \\ 1.4 & -0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -7 \\ 7 & -2.5 \\ -0.113 & 0.0974 \end{bmatrix} \tag{64}$$

Figure 4 plots the 6 elements of $\bar{\Phi}$ in the range $0.6 \text{ cm} \leq \beta \leq 1.5$ cm. Also plotted in this figure are linear approximations to $\bar{\Phi}$ using the sensitivities calculated in Eq. (64). Of particular interest are the graphs of eigenvector elements $\bar{\phi}_{21}$ and $\bar{\phi}_{12}$. The sensitivity of these elements is entirely dependent upon the off-diagonal elements of C computed with Eq. (55). It is seen from Fig. 4 that the calculated sensitivities are highly accurate. It is also interesting to note that if the method of Ref. 4 is used, the off-diagonal elements of C are calculated to be zero.

VI. Conclusion

A new method has been shown for the calculation of eigenvector derivatives for the case of repeated eigenvalues (where the repeated eigenvalues have distinct sensitivities). This technique extends Nelson's method for eigenvector sensitivities for the real, symmetric structural eigenproblem. A simple example problem supports the accuracy of the technique.

Acknowledgments

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